MINIMAX THEOREMS AND THEIR PROOFS

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1. Introduction

We suppose that X and Y are nonempty sets and $f : X \times Y \to \mathbb{R}$. A *minimax theorem* is a theorem which asserts that, under certain conditions,

$$\min_Y \max_X f = \max_X \min_Y f.$$

The original motivation for the study of minimax theorems was, of course, Von Neumann's work on games of strategy. After a lapse of nearly ten years, generalizations of Von Neumann's original result for matrices started appearing. As time went on, these generalizations became progressively more remote from game theory, and minimax theorems started becoming objects of study in their own right. In this article, we will trace the development of minimax theorems starting from Von Neumann's original result. We will discuss infinite dimensional *bilinear* results and their connection with weak compactness. We will discuss the results for *concave-convex* functions, and their generalizations to *quasiconcave-quasiconvex* functions. We will discuss various minimax theorems in which X and Y are not assumed to be subsets of vector space. These fall naturally into three classes, *topological minimax theorems* in which various connectedness hypotheses are assumed for X, Y and f, *quantitative minimax theorems* in which no special properties are assumed for X and Y, but various quantitative properties are assumed for f and, finally, *mixed minimax theorems* in which the quantitative and the topological properties are mixed. Recent developments have included *unifying metaminimax theorems*, theorems which imply simultaneously the minimax theorems of all the above three types. These latter results would tend to indicate that our initial classification of minimax theorems is too rigid. We have kept it, however, for historical reasons. We will also discuss *minimax inequalities for two or more functions.*

To a certain extent, a survey like this will always reflect the interests (prejudices) of the author. For instance, we will not discuss the kind of "local minimax theorem" that uses arguments related to the Palais-Smale condition and Ekeland's variational principle. We will also not discuss computational methods for solving games – we refer the reader to the 1974 survey article [131] by Yanovskaya. That article also contains a discussion of various infinite dimensional games. Yanovskaya's survey work was continued by Irle [34] in 1981. We would like to acknowledge that we have used both [131] and [34], as well as the many papers of Kindler, as sources

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for the history of the development of the subject. While on the subject of general references, we should mention that, in addition to his penetrating study of optimal decision rules, Aubin has a very complete bibliography in his book [1], and that there is a section on minimax theorems in the book [3] by Barbu-Precupanu.

Finally, we would like to express our sincere thanks to Heinz König for some very insightful comments on a preliminary version of this paper.

For simplicity, we shall assume that all topological spaces are Hausdorff.

2. The First Minimax Theorem

The first minimax theorem was proved by von Neumann in 1928 using topological arguments:

Theorem 1 ([124]) Let A be an $m \times n$ matrix, and X and Y be the sets of nonnegative row and column vectors with unit sum. Let $f(x, y) := xAy$. Then

$$\min_Y \max_X f = \max_X \min_Y f.$$

In this case f is a jointly continuous function of x and y, X and Y are finite dimensional simplexes, and f is bilinear. In 1938, Ville [123] gave the first elementary proof of Theorem 1, using the *Theorem of the alternative* for matrices. It is Ville's proof that von Neumann-Morgenstern expounded in [126]. Another elementary proof of Theorem 1 was given by Weyl [129] in 1950. Karlin gave an extensive analysis of the matrix case in [44]. Finally, Berge gave a proof of Theorem 1 in [5] using his theory of *regular nonlinear convexity.*

3. Infinite Dimensional Bilinear Results

Ville [123] in 1938 and Wald [127] in 1945 considered generalizations of Theorem 1 in which the function f is still bilinear, but defined on infinite-dimensional sets. (Wald's interest was motivated by his work on statistical decision functions - see [128].) This line of investigation was pursued further by Karlin [42]-[43] ([43] also contains some results on non-bilinear games), Dulmage-Peck [12] and Tjoe-Tie [120]. In 1952, Kneser [62] gave a very simple proof of a bilinear result in which X and Y are nonempty convex subsets of vector spaces, X does not even have to be topologized, Y is required to be compact and the only topological condition that is needed is that f be lower semicontinuous on Y. Kneser's result was extended by by Peck-Dulmage [97]. Wald's result was extended by Young [132], with an axiomatic theory of games. This discussion is continued in the sections **Quantitative minimax theorems** and **Minimax theorems and weak compactness**.

4. Minimax Theorems When X and Y Are More General Convex Sets

It will be convenient at this point to give a more compact notation for the various "level sets" associated with f. If $\lambda \in \mathbb{R}$ and $W \subset X$ we define

$$LE(W, \lambda) := \bigcap_{x \in W} \{y : y \in Y, f(x, y) \leq \lambda\}$$

and

$$LT(W,\lambda) := \bigcap_{x \in W} \{y : y \in Y, f(x,y) < \lambda\}.$$

If $x \in X$ we write (by abuse of notation)

$$LE(x,\lambda) = \{y : y \in Y, f(x,y) \leq \lambda\}$$

and

$$LT(x,\lambda) = \{y : y \in Y, f(x,y) < \lambda\}.$$

If $\lambda \in \mathbb{R}$ and $V \subset Y$ we define

$$GE(\lambda,V) := \bigcap_{y \in V} \{x : x \in X, f(x,y) \geq \lambda\}$$

and

$$GT(\lambda,V) := \bigcap_{y \in V} \{x : x \in X, f(x,y) > \lambda\}.$$

If $y \in Y$ we write (by abuse of notation)

$$GE(\lambda,y) = \{x : x \in X, f(x,y) \geq \lambda\}$$

and

$$GT(\lambda,y) = \{x : x \in X, f(x,y) > \lambda\}.$$

In 1937, the result of [123] was extended by von Neumann as follows:

Theorem 2 ([125]) Let X and Y be nonempty compact, convex subsets of Euclidean space, and f be jointly continuous. Suppose that f is *quasiconcave* on X, that is to say,

$$\text{for all } y \in Y \text{ and } \lambda \in \mathbb{R}, GE(\lambda,y) \text{ is convex}$$

and f is *quasiconvex* on Y, that is to say,

$$\text{for all } x \in X \text{ and } \lambda \in \mathbb{R}, LE(x,\lambda) \text{ is convex.}$$

Then

$$\min_Y \max_X f = \max_X \min_Y f.$$

In 1941, Kakutani [41] analyzed von Neumann's proof and, as a result, discovered the fixed-point theorem that bears his name. In 1952, Fan [13] generalized Theorem 2 to the case when X and Y are compact, convex subsets of (infinite dimensional) locally convex spaces and the quasiconcave and quasiconvex conditions are somewhat relaxed, while Nikaidô [92], using Brouwer's fixed-point theorem directly, generalized the same result to the case when X and Y are nonempty compact, convex subsets of (not necessarily locally convex) topological vector spaces and f is only required to be separately continuous. Nikaidô also showed in [93] that, if we replace the words *quasiconcave* and *quasiconvex* by *concave* and *convex*, then it is possible to give a proof of the minimax theorem by elementary calculus. Likewise, Moreau [87] showed that it is possible to give a proof using Fenchel duality. In 1980, Joó [37] gave a proof based on the properties of level sets, and then pointed out in [38] the connections between the level set technique and the Hahn-Banach theorem. In fact, the techniques of [37] and [38] really belonged in a much more general context. See the section **Topological minimax theorems**.

Motivated by problems in optimization theory and the theory of Lagrangians, Rockafellar [101] developed in 1970 a calculus of (possibly infinite valued) concave-convex functions in finite dimensional situations, which he extended in [102] to the infinite dimensional case. In these two paper, Rockafellar considered the concepts usually associated with convex analysis such as subdifferentials, duality and monotone operators.

5. Minimax Theorems for Separately Semicontinuous Functions

Sion, using the lemma of Knaster, Kuratowski and Mazurkiewicz on closed subsets of a finite dimensional simplex, proved the following *quasiconcave-quasiconvex u.s.c-l.s.c* result in 1958:

Theorem 3 ([114]) Let X be a convex subset of a linear topological space, Y be a compact convex subset of a linear topological space, and $f : X \times Y \to \mathbb{R}$ be upper semicontinuous on X and lower semicontinuous on Y. Suppose that,

$$\text{for all } y \in Y \text{ and } \lambda \in \mathbb{R},\ GE(\lambda, y) \text{ is convex}$$

and,

$$\text{for all } x \in X \text{ and } \lambda \in \mathbb{R},\ LE(x, \lambda) \text{ is convex.}$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

By abuse of terminology, we will continue to describe this kind of result as a *minimax theorem*. The importance of Sion's weakening of continuity to semicontinuity was that it indicated that many kind of minimax problems had equivalent formulations in terms of subsets of $X \times Y$, and led to Fan's 1972 work [16] on sets with convex sections and minimax inequalities, which have since found many applications in economic theory. (Now we would phrase some of these results in terms of *coincidence theorems* for multifunctions.) Ha [29] proved a result similar to Theorem 3, assuming instead that f was lower semicontinuous on $X \times Y$. Brézis-Nirenberg-Stampacchia [8], Kindler [46], Ha [30] and Hartung [32] showed that it was possible to relax somewhat the compactness condition in Theorem 3. It was believed for some time that Brouwer's fixed-point theorem (or the related Knaster-Kuratowski-Mazurkiewicz lemma) was required to order to prove Theorem 3. In 1966, Ghouila-Houri [22] showed that one could prove Theorem 3 using a simple combinatorial property of convex sets in finite dimensional space. Another very simple proof of Theorem 3 was given in 1988 by Komiya [63]. Using the concept of a *quarter continuous* multifunction, Komiya [65] subsequently showed that, despite the fact that the semicontinuities in Sion's result and Ha's result are different, there is a result that simultaneously generalizes both of them. See the section **Unifying**

metaminimax theorems for more recent and abstract applications of the concept of a quarter continuous multifunction.

In [84], McLinden investigated minimax results in which X and Y are not compact and f takes infinite values, using the concept of *closed saddle function* introduced by Rockafellar in [101] and [102], and also the concept of an *ε-minimax solution*. McLinden also investigated the connection between ε-minimax theorems and Ekeland's variational principle in [83]. The investigation of this type of minimax theorem appropriate for the discussion of Lagrangians was continued by Gwinner-Jeyakumar in [27] and Gwinner-Oettli in [28]. In [86], Mertens investigates the general problem of minimax theorems for separately semicontinuous functions. In this paper, he also discusses some measure-theoretic results. There is also a discussion of related results in the paper [99] by Pomerol, which has a very complete bibliography.

6. Topological Minimax Theorems

It was also believed at a more general level that proofs of minimax theorems required either the machinery of algebraic topology, or the machinery of convexity. However, in 1959, Wu, motivated by Sion's result, had already initiated research in a new direction by proving the first minimax theorem in which the conditions of convexity were totally replaced by conditions related to connectedness. Wu's paper, which appeared only a year after Sion's, evidently did not received very wide circulation in the west.

Theorem 4 ([130]) Let X be a topological space, Y be a compact separable topological space, and $f : X \times Y \to \mathbb{R}$ be separately continuous. Suppose that, for all $x_0, x_1 \in X$, there exists a continuous map $h : [0,1] \to X$ such that $h(0) = x_0, h(1) = x_1$ and,

$$\text{for all } y \in Y \text{ and } \lambda \in \mathbb{R}, \{t : t \in [0,1], f(h(t), y\} \geq \lambda\} \text{ is connected in } [0,1].$$

Suppose also that,

$$\text{for all nonempty finite subsets } W \text{ of } X \text{ and } \lambda \in \mathbb{R},\ LT(W, \lambda) \text{ is connected in } Y.$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

In 1974, Tuy removed the restrictive topological assumptions in Theorem 4, and proved the following result that generalized both Theorem 3 and Theorem 4.

Theorem 5 ([121] and [122]) Let X be a topological space, Y be a compact topological space, and $f : X \times Y \to \mathbb{R}$ be upper semicontinuous on X and lower semicontinuous on Y. Suppose that there exists a sequence $\{\lambda_s\}_{s \geq 1}$ such that $\lambda_s \downarrow \sup_X \min_Y f$ and, for all $s \geq 1$ and $x_0, x_1 \in X$, there exists a continuous map $h : [0,1] \to X$ such that $h(0) = x_0, h(1) = x_1$,

$$(b \in [a,c] \subset [0,1], y \in Y \text{ and } f(h(b), y) \leq \lambda_s) \Longrightarrow (f(h(a), y) \leq \lambda_s \text{ or } f(h(c), y) \leq \lambda_s).$$

Suppose also that,

for all $s \geq 1$ and nonempty finite subsets W of X, $LE(W, \lambda_s)$ is connected in Y.

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

A different result that generalizes both Theorem 3 and Theorem 4 was given in 1984 by Geraghty-Lin:

Theorem 6 ([19]) Let X be a topological space, Y be a compact topological space, and $f : X \times Y \to \mathbb{R}$ be lower semicontinuous on X and also lower semicontinuous on Y. Suppose that, for all $x_0, x_1 \in X$, there exists a continuous map $h : [0,1] \to X$ such that $h(0) = x_0, h(1) = x_1$, and

$$b \in [a,c] \subset [0,1] \Longrightarrow f(h(b),.) \geq f(h(a),.) \wedge f(h(c),.) \text{ on } Y,$$

where "$\wedge$" stands for "minimum". Suppose also that,

for all nonempty finite subsets W of X and $\lambda \in \mathbb{R}$, $LE(W, \lambda)$ is connected in Y.

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

After analyzing the paper [37] of Joó already mentioned, Stachó introduced in 1980 the concept of an *interval space*, a topological space X such that, for all $x_1, x_2 \in X$, there exists a connected subset $[x_1, x_2]$ of X such that $[x_1, x_2] = [x_2, x_1] \supset \{x_1, x_2\}$. A subset C of X is called *interval convex* if $x_1, x_2 \in C \Longrightarrow [x_1, x_2] \subset C$. Stachó then established the following result:

Theorem 7 ([115]) Let X and Y be compact interval spaces, and $f : X \times Y \to \mathbb{R}$ be continuous. Suppose that,

for all $y \in Y$ and $\lambda \in \mathbb{R}$, $GE(\lambda, y)$ is interval convex

and,

for all $x \in X$ and $\lambda \in \mathbb{R}$, $LE(x, \lambda)$ is interval convex.

Then

$$\min_Y \max_X f = \max_X \min_Y f.$$

Komornik subsequently proved in [66] a minimax theorem for interval spaces which generalized both Theorem 7 and the result [29] of Ha already mentioned.

Stachó also introduced the concept of a *Dedekind complete* interval space and proved a second minimax theorem, which generalizes Theorem 3. Here is a slightly simplified (see below) version of this second result:

Theorem 8 ([115]) Let X be a Hausdorff, Dedekind complete interval space, Y be a compact interval space, and $f : X \times Y \to \mathbb{R}$ be upper semicontinuous on X and lower semicontinuous on Y. Suppose that,

$$\text{for all } y \in Y \text{ and } \lambda \in \mathbb{R},\ GE(\lambda, y) \text{ is interval convex}$$

and,

$$\text{for all } x \in X \text{ and } \lambda \in \mathbb{R},\ LE(x, \lambda) \text{ is interval convex.}$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

In 1989, Kindler-Trost established a very general topological minimax theorem. Here is a slightly simplified (see below) version of their result, which contains both Theorem 5 and Theorem 8.

Theorem 9 ([61]) Let X be an interval space, Y be a compact topological space, and $f : X \times Y \to \mathbb{R}$ be upper semicontinuous on X and lower semicontinuous on Y. Suppose that there exists a sequence $\{\lambda_s\}_{s \geq 1}$ such that $\lambda_s > \sup_X \min_Y f$, $\lambda_s \to \sup_X \min_Y f$ and,

$$\text{for all } s \geq 1 \text{ and } y \in Y,\ GT(\lambda_s, y) \text{ is interval convex.}$$

Suppose also that,

$$\text{for all } s \geq 1 \text{ and nonempty finite subsets } W \text{ of } X,\ LE(W, \lambda_s) \text{ is connected in } Y.$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

In fact, the semicontinuity conditions and the compactness of Y assumed in Theorem 8 and Theorem 9 are stronger than the topological conditions actually assumed in [115] and [61]. We have adopted these simplifications so as not to overburden the reader with too many technicalities, and also to achieve a certain unity of presentation.

The above results are all subsumed by the following general topological minimax theorem established by König. Again, we have simplified the statement somewhat.

Theorem 10 ([72]) Let X be a connected topological space, Y be a compact connected topological space, and $f : X \times Y \to \mathbb{R}$ be upper semicontinuous on X and lower semicontinuous on Y. Suppose that, for all $\lambda > \sup_X \min_Y f$ **either**

$$\text{for all nonempty subsets } V \text{ of } Y,\ GT(\lambda, V) \text{ is connected in } X, \text{ and}$$
$$\text{for all nonempty finite subsets } W \text{ of } X,\ LE(W, \lambda) \text{ is connected in } Y,$$

or

$$\text{for all nonempty subsets } V \text{ of } Y,\ GT(\lambda, V) \text{ is connected in } X, \text{ and}$$
$$\text{for all nonempty finite subsets } W \text{ of } X,\ LT(W, \lambda) \text{ is connected in } Y,$$

or

for all nonempty subsets V of Y, $GE(\lambda, V)$ is connected in X, and
for all nonempty finite subsets W of X, $LT(W, \lambda)$ is connected in Y,

or

for all nonempty subsets V of Y, $GE(\lambda, V)$ is connected in X, and
for all nonempty finite subsets W of X, $LE(W, \lambda)$ is connected in Y.

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

There is also in [72] a result similar to Theorem 10 with different semicontinuity assumptions. We note the basic asymmetry in the above results: in one variable we allow arbitrary intersections, while in the other variable we only allow finite intersections. König [73] has recently given an example showing the failure of the "symmetric" theorem in which we allow only finite intersections in both variables.

In [33], Horvath proved a result similar to Theorem 6 only with X a convex set in some vector space, and the topology of X replaced by the natural topology of all the line segments in X. More results in the direction of Theorem 9 were proved by Ricceri in [100].

This discussion will be continued in the section **Unifying metaminimax theorems.**

7. Quantitative Minimax Theorems

In 1953, Fan was the first person to take the theory of minimax theorems out of the context of convex subsets of vector spaces when he established the following result generalizing [62]:

Theorem 11 ([14]) Let X be a nonempty set and Y be a nonempty compact topological space. Let $f : X \times Y \to \mathbb{R}$ be lower semicontinuous on Y. Suppose that f is *concavelike* on X and *convexlike* on Y, that is to say:

for all $x_1, x_2 \in X$ and $\alpha \in [0,1]$, there exists $x_3 \in X$ such that
$$f(x_3, .) \geq \alpha f(x_1, .) + (1-\alpha) f(x_2, .) \text{ on } Y,$$

and

for all $y_1, y_2 \in Y$ and $\beta \in [0,1]$, there exists $y_3 \in Y$ such that
$$f(., y_3) \leq \beta f(., y_1) + (1-\beta) f(., y_2) \text{ on } X.$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

See Parthasarathy [96] for further developments in this direction. In [14], Fan also proved a minimax theorem for almost periodic functions of two variables, which was subsequently generalized by Tjoe-The [120] and Parthasarathy [95]. Aubin ([1] and [2]) proved results related to Theorem 11, in terms of the concepts of γ-vexity. In [6], Borwein and Zhuang give a very short proof of Theorem 11 using the Eidelheit separation theorem. König in 1968, and then Simons in 1971, proved the following result generalizing Theorem 11:

Theorem 12 ([67] and [105]) Let X be a nonempty set and Y be a nonempty compact topological space. Let $f : X \times Y \to \mathbb{R}$ be lower semicontinuous on Y. Suppose that:

$$\text{for all } x_1, x_2 \in X, \text{ there exists } x_3 \in X \text{ such that } f(x_3,.) \geq \frac{f(x_1,.) + f(x_2,.)}{2} \text{ on } Y,$$

and,

$$\text{for all } y_1, y_2 \in Y, \text{ there exists } y_3 \in Y \text{ such that } f(.,y_3) \leq \frac{f(.,y_1) + f(.,y_2)}{2} \text{ on } X.$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

At first sight, the difference between Theorem 11 and Theorem 12 is not very striking. The proofs in [67] and [105] both used a version of the Hahn-Banach theorem due to Mazur-Orlicz. However, both proofs followed the same pattern as that of [14], replacing the convexity of the the sets X and Y by statements about the convexity of the functional values of f. It turned out subsequently that the difference between Theorem 11 and Theorem 12 was quite significant, and led eventually, *via* the steps that will be described below, to the *unifying metaminimax theorems* to be discussed in a later section.

Since the Mazur-Orlicz theorem is itself a very special kind of minimax theorem, and is not as well known as it deserves to be, it seems appropriate for us to take a small digression and give a few additional details. First, here is a statement of the result itself:

Theorem 13 ([85]) Let E be a real vector space, $S : E \to \mathbb{R}$ be sublinear and C be a nonempty convex subset of E. Then there exists a linear functional L on E such that

$$L \leq S \text{ on } E \text{ and } \inf_C L = \inf_C S.$$

Other applications of the Mazur-Orlicz theorem and related *sandwich theorems* are discussed by König in [68], [69] and [70], and Neumann in [89], [90] and [91]. These include not only numerous applications to measure theory and Hardy algebra theory, but also to the theory of flows in infinite networks. See also the paper [99] by Pomerol already mentioned in a previous section. We now return to our historical narrative. In 1977, Neumann proved the following generalization of Theorem 12:

Theorem 14 ([88] and [74]) Let X be a nonempty set and Y be a nonempty compact topological space. Let $f : X \times Y \to \mathbb{R}$ be lower semicontinuous on Y. Suppose that there exists $\alpha \in (0,1)$ such that,

$$\text{for all } x_1, x_2 \in X, \text{ there exists } x_3 \in X \text{ such that}$$
$$f(x_3,.) \geq \alpha f(x_1,.) + (1-\alpha) f(x_2,.) \text{ on } Y,$$

and there exists $\beta \in (0,1)$ such that,

$$\text{for all } y_1, y_2 \in Y, \text{ there exists } y_3 \in Y \text{ such that}$$
$$f(.,y_3) \leq \beta f(.,y_1) + (1-\beta) f(.,y_2) \text{ on } X.$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

Actually, Neumann proved a result that was more general that this by a factor of ε, but we will not discuss these technicalities in this article. The result of [88] was subsequently extended by Fuchssteiner-König [17], but their results do not qualify as minimax theorems in our sense. In [48], Kindler investigated the connections between minimax theorems and the representation of integrals. Then there was a hiatus of several years in this line of research.

Activity in this area resumed with a sequence of papers in which minimax theorems were proved without recourse to arguments based ultimately on convexity. We cite the 1989 paper by Lin-Quan, who generalized Theorem 14 with the following result:

Theorem 15 ([76]) Let X be a nonempty set and Y be a nonempty compact topological space. Let $f : X \times Y \to \mathbb{R}$ be lower semicontinuous on Y. Suppose there exists $\alpha \in (0,1)$ such that,

$$\left.\begin{array}{c}\text{for all } x_1, x_2 \in X, \text{ there exists } x_3 \in X \text{ such that}\\ f(x_3,.) \geq \alpha[f(x_1,.) \vee f(x_2,.)] + (1-\alpha)[f(x_1,.) \wedge f(x_2,.)] \text{ on } Y,\end{array}\right\} \tag{15.1}$$

and there exists $\beta \in (0,1)$ such that,

$$\text{for all } y_1, y_2 \in Y, \text{ there exists } y_3 \in Y \text{ such that}$$
$$f(.,y_3) \leq \beta[f(.,y_1) \vee f(.,y_2)] + (1-\beta)[f(.,y_1) \wedge f(.,y_2)] \text{ on } X,$$

where "$\vee$" stands for "maximum". Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

There was actually a slightly earlier (1985) more general result by Irle which, unfortunately, did not receive much circulation. Here we present a slightly simplified version of Irle's result. Irle defines an *averaging function* to be a continuous function $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that ϕ is nondecreasing in each variable, $\phi(\lambda,\lambda) = \lambda$,

$$\text{if } \lambda \neq \mu \text{ then } \lambda \wedge \mu < \phi(\lambda,\mu) < \lambda \vee \mu,$$

and then proves:

Theorem 16 ([35]) Let X be a nonempty set and Y be a nonempty compact topological space. Let $f : X \times Y \to \mathbb{R}$ be lower semicontinuous on Y. Suppose there exist averaging functions ϕ and ψ such that,

$$\text{for all } x_1, x_2 \in X, \text{ there exists } x_3 \in X \text{ such that}$$

$$f(x_3, .) \geq \phi(f(x_1, .), f(x_2, .)) \text{ on } Y,$$

and,

$$\text{for all } y_1, y_2 \in Y, \text{ there exists } y_3 \in Y \text{ such that}$$

$$f(., y_3) \leq \psi(f(., y_1), f(., y_2)) \text{ on } X.$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

Irle [36] has given an application of the above result to *hide-and-seek games*. In 1990, Kindler [54] gave a generalization of Theorem 16 using the concept of a *mean function*, which is too complicated to describe in detail here. Averaging functions in the sense of Irle are mean functions, but mean functions do not have the restriction of being continuous that averaging functions have. Kindler's paper [54] has a deeper significance which we will return to in the section **Unifying metaminimax theorems**. In 1990, Simons gave another result that extends Theorem 15:

Theorem 17 ([109]) Let X be a nonempty set and Y be a nonempty compact topological space. Let $f : X \times Y \to \mathbb{R}$ be lower semicontinuous on Y. Suppose that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$\text{for all } x_1, x_2 \in X, \text{ there exists } x_3 \in X \text{ such that } f(x_3, .) \geq f(x_1, .) \wedge f(x_2, .) \text{ on } Y \tag{17.1}$$

and

$$y \in Y \text{ and } |f(x_1, y) - f(x_2, y)| \geq \varepsilon \Longrightarrow f(x_3, y) \geq f(x_1, y) \wedge f(x_2, y) + \delta, \tag{17.2}$$

and

$$\text{for all } y_1, y_2 \in Y, \text{ there exists } y_3 \in Y \text{ such that } f(., y_3) \leq f(., y_1) \vee f(., y_2) \text{ on } X \tag{17.3}$$

and

$$x \in X \text{ and } |f(x, y_1) - f(x, y_2)| \geq \varepsilon \Longrightarrow f(x, y_3) \leq f(x, y_1) \vee f(x, y_2) - \delta.$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

At this point, it would be in order to explain the motivation behind the series of results discussed above. It is easy to see that, even given the strongest topological conditions, (17.1) and (17.3) are *not* sufficient to force the minimax relation to hold. Theorem 12, Theorem 14, Theorem 15 and Theorem 16 had successively weaker hypotheses, which *did* force the minimax relation to hold. Theorem 17 was another result with hypotheses weaker than Theorem 15 which also forced the minimax relation to hold. The hypotheses of *all* the results mentioned above imply (17.1) and (17.3). Theorem 16 and its generalization by Kindler both use external functions ϕ and ψ, while Theorem 17 does not. These two kinds of results were unified by Simons in [111] using the concept of a *staircase*, which is quite technical and too complicated to go into here. A deep and very penetrating study of this kind of problem was also made by König-Zartmann in [75]. Nevertheless, there is in fact a very simple combinatorial principle behind all these results, which we will discuss in the section **Unifying metaminimax theorems**. We should mention finally that Kindler has recently incorporated some of the techniques described above to obtain extensions of the results in [48] to statistical decision theory and the theory of convex metric spaces – see [59] and [60], and also the paper [98] on minimax risk by Pinelis.

8. Mixed Minimax Theorems

In 1972, Terkelsen proved the first *mixed* minimax theorem. Specifically, one of the conditions in the following result is taken from the topological Theorem 6 and the other from the quantitative Theorem 12:

Theorem 18 ([119]) Let X be a nonempty set and Y be a nonempty compact topological space. Let $f : X \times Y \to \mathbb{R}$ be lower semicontinuous on Y. Suppose that,

$$\left.\begin{array}{c}\text{for all } x_1, x_2 \in X, \text{ there exists } x_3 \in X \text{ such that} \\ f(x_3, .) \geq [f(x_1, .) + f(x_2, .)]/2 \text{ on } Y.\end{array}\right\} \tag{18.1}$$

Suppose also that,

$$\text{for all nonempty finite subsets } W \text{ of } X \text{ and } \lambda \in \mathbb{R},\ LE(W, \lambda) \text{ is connected in } Y. \tag{18.2}$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

This result was subsequently generalized by Geraghty-Lin [18] who proved in 1983 that (18.1) can be weakened to: there exists $\alpha \in (0, 1)$ such that,

$$\begin{array}{c}\text{for all } x_1, x_2 \in X, \text{ there exists } x_3 \in X \text{ such that} \\ f(x_3, .) \geq \alpha[f(x_1, .) \vee f(x_2, .)] + (1 - \alpha)[f(x_1, .) \wedge f(x_2, .)] \text{ on } Y,\end{array}$$

that is to say, (15.1) is satisfied. Simons [110] proved that this condition can be further weakened to: for all $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$\text{for all } x_1, x_2 \in X, \text{ there exists } x_3 \in X \text{ such that } f(x_3, .) \geq f(x_1, .) \wedge f(x_2, .) \text{ on } Y$$

and

$$y \in Y \text{ and } |f(x_1, y) - f(x_2, y)| \geq \varepsilon \Longrightarrow f(x_3, y) \geq f(x_1, y) \wedge f(x_2, y) + \delta,$$

that is to say, (17.1) and (17.2) are satisfied. Kindler, in the paper [54] already mentioned gave another generalization of Theorem 18 using his concept of a *mean function*.

Takahashi showed in [117] that (18.2) could also be weakened somewhat. Furthermore, Takahashi-Takahashi have applied Terkelsen's methods in [118] to obtain results on fuzzy sets. Finally, Stefanescu in [116] proved a minimax theorem similar to Theorem 18 in which (18.2) was replaced by the appropriate set-theoretic assumption.

9. Unifying Metaminimax Theorems

The first hint that our classification of general minimax theorems into *topological*, *quantitative* and *mixed* might be too rigid, was probably provided in 1982 by Joó-Stachó, who showed in [40] that Theorem 11, which we have classified as *quantitative*, could be deduced using Radon measures from the result [8] of Brézis-Nirenberg-Stampacchia already mentioned which could, in turn, be deduced from Theorem 8, which we have classified as *topological*. A second hint was provided in 1985 and 1986 by Geraghty-Lin, who investigated in [20] and [21] a continuum of minimax theorems joining Theorem 4, which we have classified as *topological*, and Theorem 18, which we have classified as *mixed*. A third hint was provided in 1989 by Komiya, who proved in [64] a result which contained both Theorem 11 and also [19], which we have classified as *topological*. It was Kindler in [54] who first realized in 1990 that some concept akin to connectedness might be involved in minimax theorems where the topological condition of connectedness was not explicitly assumed. This idea was pursued by Simons with the introduction in 1992 of the concept of *pseudoconnectedness*. We say that sets H_0 and H_1 are *joined* by a set H if

$$H \subset H_0 \cup H_1, H \cap H_0 \neq \emptyset \text{ and } H \cap H_1 \neq \emptyset.$$

We say that a family $\mathcal{H}$ of sets is *pseudoconnected* if:

$$\text{if } H_0, H_1, H \in \mathcal{H} \text{ and } H_0 \text{ and } H_1 \text{are joined by } H \text{ then } H_0 \cap H_1 \neq \emptyset.$$

Any family of closed connected subsets of a topological space is pseudoconnected. So also is any family of open connected subsets. Our next result is an improvement suggested by some comments of Heinz König of a result from [112] and [113].

We shall say that a subset W of X is *good* if W is finite and,

$$\text{for all } x \in X,\ LE(x, \sup_X \inf_Y f) \cap LE(W, \sup_X \inf_Y f) \neq \emptyset.$$

Theorem 19 Let Y be a topological space, and Λ be a nonempty subset of $\mathbb{R}$ such that $\inf \Lambda = \sup_X \inf_Y f$. Suppose that, for all $\lambda \in \Lambda$ and good subsets W of X,

$$\text{for all } x \in X, LE(x, \lambda) \text{ is closed and compact,} \tag{19.1}$$

$$\{LE(x, \lambda) \cap LE(W, \lambda)\}_{x \in X} \text{ is pseudoconnected} \tag{19.2}$$

and, for all $x_0, x_1 \in X$, there exists $x \in X$ such that

$$LE(x_0,\lambda) \text{ and } LE(x_1,\lambda) \text{ are joined by } LE(x,\lambda) \cap LE(W,\lambda). \tag{19.3}$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

Proof Let $x \in X$. If $\mu \in \mathbb{R}$ and $\mu > \sup_X \min_Y f$ then $\mu > \min f(x,Y)$, from which $LE(x,\mu) \neq \emptyset$. From (19.1) and the finite intersection property, $LE(x, \sup_X \min_Y f) \neq \emptyset$. Thus $\emptyset$ is good. We now prove by induction that all finite subsets of X are good. So suppose that $n \geq 1$ and

$$W \subset X \text{ and } \operatorname{card} W \leq n-1 \Longrightarrow W \text{ is good.} \tag{19.4}$$

Let $V \subset X$ and $\operatorname{card} V = n$. Let $x_0 \in V$ and set $W := V\backslash\{x_0\}$. From the induction hypothesis (19.4), W is good. Let $x_1 \in X$ be arbitrary. Let $\lambda \in \Lambda$ be arbitrary. From (19.3), there exists $x \in X$ such that $LE(x_0,\lambda)$ and $LE(x_1,\lambda)$ are joined by $LE(x,\lambda) \cap LE(W,\lambda)$. Equivalently,

$$LE(x_0,\lambda) \cap LE(W,\lambda) \text{ and } LE(x_1,\lambda) \cap LE(W,\lambda) \text{ are joined by } LE(x,\lambda) \cap LE(W,\lambda).$$

From (19.2), $LE(x_0,\lambda) \cap LE(x_1,\lambda) \cap LE(W,\lambda) \neq \emptyset$, that is to say, $LE(x_1,\lambda) \cap LE(V,\lambda) \neq \emptyset$. Since this holds for all $\lambda \in \Lambda$, from (19.1) and the finite intersection property again,

$$LE(x_1, \sup_X \min_Y f) \cap LE(V, \sup_X \min_Y f) \neq \emptyset.$$

Since this is valid for all $x_1 \in X$, V is good. This completes the inductive step of the proof that all finite subsets of X are good. It now follows from (19.1) and the finite intersection property for a third time that $LE(X, \sup_X \min_Y f) \neq \emptyset$. This completes the proof of Theorem 19. Given the obvious topological motivation behind the concept of pseudoconnectedness, it is hardly surprising that Theorem 19 implies all the results mentioned in the section **Topological minimax theorems** (except for some of the parts of Theorem 10). What is more unexpected is that Theorem 19 implies all the results mentioned in the sections **Quantitative minimax theorems** and **Mixed minimax theorems** (except possibly that of [116]), as well as some new results. On the other hand, as we have seen above the proof of Theorem 19 is certainly not profound — the real work is done in proving that the conditions (19.2) and (19.3) are satisfied in any of the particular cases. That is why we prefer to describe Theorem 19 as a **metaminimax theorem** rather than a **minimax theorem**, it is really a device for obtaining minimax theorems rather than a minimax theorem in its own right. Another way of looking at Theorem 19 is as a "decomposition of the minimax property". This avenue is pursued quite profoundly in the paper [75] by König-Zartmann (already mentioned in the section **Quantitative minimax theorems**).

The remainder of this section is devoted to some results which are at the interface between minimax theory and abstract set theory. Since many of them are quite technical, we will not go into them in great detail. Most of results discussed below were motivated by Theorem 19, [75], and Theorem 10.

Suppose that $\{C_x\}_{x\in X}$ is a family of subsets of Y. If $y \in Y$, define the *conjugate set* by

$$C_y^* := \{x : x \in X,\ y \notin C_x\}$$

Kindler proved in [55] that the sets $\{C_x\}_{x\in X}$ have the finite intersection property if, and *only if*, there exist there exist topologies on X and Y such that

$$Y \text{ is compact,}$$
$$\text{all the sets } C_x \text{ are closed in } Y,$$
$$\text{for each closed subset } F \text{ of } Y,\ \bigcup\{C_y^* : y \in F\} \text{ is open in } X,$$
$$\text{all finite intersections of the sets } C_x \text{ are connected in } Y,$$

and

$$\text{all intersections of the sets } C_y^* \text{ are connected in } X.$$

Kindler deduced a number of minimax theorems from this observation. In [56], he gave necessary *and sufficient* conditions in terms of these and allied concepts that the minimax relation hold. In [57], motivated by the *interval spaces* introduced by Stachó in [115], Kindler considered a *midset space*, which is simply a set S and a function $S \times S \to 2^S$, and went on in [58] to use this concept and the concept of *quarter continuous multifunction* introduced by Komiya in [65] to prove a generalization of Theorem 3.

10. Connections with Weak Compactness

In 1971, Simons proved that there are limitations on the extent to which one can generalize the minimax theorem. Specifically:

Theorem 20 ([105]) Suppose that X is a nonempty bounded, convex, complete subset of a locally convex space E with dual space E^*, and

$$\inf_{y\in Y}\sup_{x\in X}\langle x,y\rangle = \sup_{x\in X}\inf_{y\in Y}\langle x,y\rangle$$

whenever Y is a nonempty convex, equicontinuous, subset of E^*. Then

$$X \text{ is weakly compact.}$$

Coupled with the following result, one can obtain a proof of R. C. James's *sup theorem*:

Theorem 21 ([106]) . If X is a nonempty bounded, convex subset of a locally convex space E such that every element of the dual space E^* attains its supremum on X, and Y is any nonempty convex equicontinuous subset of E^*, then

$$\inf_{y\in Y}\sup_{x\in X}\langle x,y\rangle = \sup_{x\in X}\inf_{y\in Y}\langle x,y\rangle.$$

In 1974, using the concept of *ordered iterated limits*, De Wilde [10] simplified and extended the result of Theorem 21. Further work on this topic was also done by Ha in [31]. In [103] and [104], Rodé introduced the related theory of *superconvexity*, an axiomatic theory of infinite convex combinations. See the articles [68], [69] and [71] by König for later developments in this direction.

One of the aspects of von Neumann's original minimax theorem that we have not mentioned explicitly is the idea of extending a game from a finite set of *pure* strategies to a convex set of *mixed* strategies. What Theorem 1 showed is that, even if there is no saddle point for the original game, there is always one for the extended game. Indeed, many of the results mentioned in the section **Infinite dimensional bilinear results** were motivated by the problems involved in extending a game. Generalizing a result from the paper [132] of Young already mentioned, Kindler established the following in 1976:

Theorem 22 ([45]) Let X, Y be nonempty sets and $a : X \times Y \to \mathbb{R}$ be bounded. Suppose that

$$\lim_{n\to\infty}\lim_{m\to\infty} a(x_m, y_n) \leq \lim_{m\to\infty}\lim_{n\to\infty} a(x_m, y_n)$$

whenever $\{x_m\}_{m\geq 1}$ and $\{y_n\}_{n\geq 1}$ are sequences in X and Y, respectively, such that the iterated limits exist. Then

$$\inf_{\nu\in P(Y)}\sup_{\mu\in P(X)}\int_X\int_Y a\,d\nu d\mu = \sup_{\mu\in P(X)}\inf_{\nu\in P(Y)}\int_X\int_Y a\,d\nu d\mu,$$

where $P(S)$ is the set of all probability measures on S with finite support.

The hypothesis on a in Theorem 22 is exactly the condition on *ordered iterated limits* introduced by De Wilde in [10]. In [49], Kindler combined Theorem 12 and Theorem 22 to obtain results on the extension of games, which led to simple proofs of the Krein-Smulian and Eberlein-Smulian theorems. This shows again the close connection between minimax theorems and weak compactness. In [50] and [51], Kindler considers generalizations of Theorem 22 to the case when μ and ν are allowed to be more general finitely additive measures, and the connections with other concepts related to weak compactness, such as Fubini's theorem for finitely additive measures, and Pták's combinatorial lemma. Kindler then showed that the connection between ordered iterated limits and minimax theorems was even tighter with a two-function generalization of Theorem 22. Here is a slightly simplified version of Kindler's result:

Theorem 23 ([52]) Let X, Y be nonempty sets and $a, b : X \times Y \to \mathbb{R}$ be bounded. Then:

$$\lim_{n\to\infty}\lim_{m\to\infty} a(x_m, y_n) \leq \lim_{m\to\infty}\lim_{n\to\infty} b(x_m, y_n)$$

whenever $\{x_m\}_{m\geq 1}$ and $\{y_n\}_{n\geq 1}$ are sequences in X and Y, respectively, such that the iterated limits exist,

if, and only if,

$$\inf_{\nu\in P(T)}\sup_{\mu\in P(S)}\int_S\int_T a\,d\nu d\mu \leq \sup_{\mu\in P(S)}\inf_{\nu\in P(T)}\int_S\int_T b\,d\nu d\mu$$

whenever S is a nonempty subset of X and T is a nonempty subset of Y.

11. Minimax Inequalities for Two or More Functions

Motivated by Nash equilibrium and the theory of non-cooperative games, Fan generalized Theorem 3 in 1964 to the case of more than one function. In particular, he proved the following two-function minimax inequality:

Theorem 24 ([15]) Let X and Y be nonempty compact, convex subsets of topological vector spaces and $f, g : X \times Y \to \mathbb{R}$. Suppose that f is lower semicontinuous on Y,

$$\text{for all } y \in Y \text{ and } \lambda \in \mathbb{R}, \{x : x \in X, f(x,y) \geq \lambda\} \text{ is convex,}$$

$$\text{for all } x \in X \text{ and } \lambda \in \mathbb{R}, \{y : y \in Y, g(x,y) \leq \lambda\} \text{ is convex,}$$

g is upper semicontinuous on X, and

$$f \leq g \text{ on } X \times Y.$$

Then

$$\min_Y \sup_X f \leq \sup_X \inf_Y g.$$

Liu [81] observed that Theorem 24 is true even if X is not assumed to be compact, and that Theorem 24 actually unifies the theory of minimax theorems and the theory of variational inequalities. Theorem 24 was extended even further by Ben-El-Mechaiekh, Deguire and Granas who proved:

Theorem 25 ([4]) Let X and Y be nonempty compact, convex subsets of topological vector spaces and $f, s, t, g : X \times Y \to \mathbb{R}$. Suppose that f is lower semicontinuous on Y,

$$\text{for all } y \in Y \text{ and } \lambda \in \mathbb{R}, \{x : x \in X, s(x,y) \geq \lambda\} \text{ is convex,}$$

$$\text{for all } x \in X \text{ and } \lambda \in \mathbb{R}, \{y : y \in Y, t(x,y) \leq \lambda\} \text{ is convex,}$$

g is upper semicontinuous on X, and

$$f \leq s \leq t \leq g \text{ on } X \times Y.$$

Then

$$\inf_Y \sup_X f \leq \sup_X \inf_Y g.$$

In [23], Granas-Liu extended Ha's result of [30] to two and three functions In 1981, Fan (unpublished) and Simons generalized Theorem 12 by proving the following two-function minimax inequality:

Theorem 26 ([107]) Let X be a nonempty set, Y be a compact topological space and $f, g : X \times Y \to \mathbb{R}$. Suppose that f is lower semicontinuous on Y,

$$\text{for all } y_1, y_2 \in Y, \text{ there exists } y_3 \in Y \text{ such that } f(.,y_3) \leq \frac{f(.,y_1) + f(.,y_2)}{2} \text{ on } X,$$

$$\text{for all } x_1, x_2 \in X, \text{ there exists } x_3 \in X \text{ such that } g(x_3,.) \geq \frac{g(x_1,.) + g(x_2,.)}{2} \text{ on } Y,$$

and

$$f \le g \text{ on } X \times Y.$$

Then

$$\min_Y \sup_X f \le \sup_X \inf_Y g.$$

Theorem 26 also unifies the theory of minimax theorems and the theory of variational inequalities. The curious feature about Theorem 24 and Theorem 26 is that they have *opposite geometric pictures*. This question is discussed in [107] and [108]. The relationship between Theorem 24 and Brouwer's fixed-point theorem is quite interesting. As we have already pointed out, Sion's minimax theorem, Theorem 3, can be proved in an elementary fashion without recourse to fixed-point related concepts. On the other hand, Theorem 24, which is a generalization of Theorem 3 can, in fact, be used to *prove* Tychonoff's fixed-point theorem. (See [15] for more details.)

In [39], Joó-Kassay defined a *pseudoconvex space* to be a topological space X with an appropriate family of continuous maps from finite dimensional simplices into certain subsets of X. They then proved that Theorem 24 can be generalized to this more abstract situation. In the same paper, they gave a counterexample showing that the "obvious" generalization of Theorem 17 to two functions fails.

On the other hand, Lin-Quan gave the following two function generalization of Theorem 15:

Theorem 27 ([77]) Let X be a nonempty set and Y be a nonempty compact topological space. Let $f, g : X \times Y \to \mathbb{R}$ be lower semicontinuous on Y. Suppose there exists $\alpha \in (0,1)$ such that,

$$\text{for all } x_1, x_2 \in X, \text{ there exists } x_3 \in X \text{ such that}$$
$$f(x_3, .) \ge \alpha[f(x_1, .) \vee g(x_2, .)] + (1-\alpha)[f(x_1, .) \wedge g(x_2, .)] \text{ on } Y,$$

and there exists $\beta \in (0,1)$ such that,

$$\text{for all } y_1, y_2 \in Y, \text{ there exists } y_3 \in Y \text{ such that}$$
$$g(., y_3) \le \beta[f(., y_1) \vee g(., y_2)] + (1-\beta)[f(., y_1) \wedge g(., y_2)] \text{ on } X,$$

and

$$f \le g \text{ on } X \times Y.$$

Then

$$\min_Y \sup_X f \le \sup_X \min_Y g.$$

By contrast with the counterexample of Joó-Kassay mentioned above, the conditions in Theorem 27 "mix up" the functional values of f and g. In [78], Lin-Quan generalize Theorem 27 using the concept of a *staircase* introduced in [111]. In [79], they gave a generalization of Komiya's topological result of [65] to two functions, while in [80] they show that the compactness condition in [79] can be relaxed.

Granas-Liu have proved a number of minimax inequalities for two or more functions. In [24], they extended Fan's result of [14] to three and four functions, and in [26] they extended the result of Theorem 26 to four functions with a weakening of the conditions of semiconvexlike and semiconcavelike, but under the assumption that X is also a compact topological space.

In [53], Kindler consolidated the results of his papers [45]-[52] and considered approximate two-function minimax inequalities.

12. Coincidence Theorems

A *coincidence theorem* is a theorem that asserts that if $S : X \to 2^Y$ and $T : Y \to 2^X$ have nonempty values and satisfy certain other conditions then there exist $x_0 \in X$ and $y_0 \in Y$ such that $y_0 \in Sx_0$ and $x_0 \in Ty_0$. The connection with minimax theorems is as follows: suppose that $\inf_Y \sup_X f \neq \sup_X \inf_Y f$. Then there exists $\lambda \in \mathbb{R}$ such that

$$\sup_X \inf_Y f < \lambda < \inf_Y \sup_X f.$$

Hence,

$$\text{for all } x \in X, \text{ there exists } y \in Y \text{ such that } f(x,y) < \lambda$$

and,

$$\text{for all } y \in Y, \text{ there exists } x \in X \text{ such that } f(x,y) > \lambda.$$

Define $S : X \to 2^Y$ and $T : Y \to 2^X$ by

$$Sx := \{y : y \in Y, f(x,y) < \lambda\} \text{ and } Tx := \{x : x \in X, f(x,y) > \lambda\}.$$

Then the values of the multifunctions S and T are nonempty. If S and T were to satisfy a coincidence theorem then we would have $x_0 \in X$ and $y_0 \in Y$ such that

$$f(x_0, y_0) < \lambda \text{ and } f(x_0, y_0) > \lambda,$$

which is clearly impossible. Thus this coincidence theorem would imply that

$$\inf_Y \sup_X f = \sup_X \inf_Y f.$$

The coincidence theorems known in algebraic topology consequently give rise to corresponding minimax theorems. The first person to have used this idea seems to have been Debreu [11] in 1952, and this line of investigation was pursued by Bourgin [7] and McClendon [82]. See also the paper [25] by Granas-Liu for analogous results for minimax inequalities involving more than one function. There is a very extensive literature of coincidence theorems. We refer the reader to the paper [94] by Park for some further pointers in this direction.

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